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The Element ideal graph$\Gamma_{p^{\alpha}q^{\beta}}(Z)$

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Abstract

Let R be acommutative ring with identity and let x be an element of R. The Element Ideal Graph $\Gamma_x(R)$ is a graph whose vertex set is the set of non-trivial ideals of R and two distinct ideal vertices I and J are adjacent if and only if $x \in IJ$. In this paper we investigate the element ideal graph $\Gamma_{p^{\alpha}q^{\beta}}(Z)$ to explore some of its properties with providing some examples, where Z is the set of integers, $\alpha, \beta \in Z^+$ and p and q are distinct prime numbers.

Introduction

The zero divisor graph of a commutative ring was first introduced by Beck in 1988 [4], and further studied in [2, 3, 9, 10]. The annihilating ideal graph AG(R) is a graph with vertex set AG*(R)=AG(R)\{(0)} such that there is an edge between vertices I and J if and only if $IJ = (0)$. The idea of annihilating ideal graph was introduced by Behboodi and Rakeei in [5,6]. In [1], the notion of the element ideal graph of a commutative ring R have been introduced, represented by $\Gamma_x(R)$, as a graph whose vertex set is the set of non-trivial ideals of R and two ideal vertices I and J are adjacent if and only if x∈IJ. And the relationship between any two of the zero divisor graph, the annihilating ideal graph and the element ideal graph have been illustrated. From now on:

1. We denote the edge of terminals I and J by $\{I, J\}$.
2. We denote The set of integers by Z and we consider $\alpha, \beta \in Z^+$.
3. We consider p and q as two distinct prime numbers.

1. *Preliminaries*

The aim of this section is to give some definitions and theorems which may be used in the sequel.

Definition 1.1[1]:

Let R be a commutative ring with identity and let x∈R. The element ideal graph is a graph whose vertex set consists of non-trivial ideals of R, for which two of its distinct ideal vertices I and J are adjacent if and only if x∈IJ. We denote the element ideal graph by $\Gamma_x(R)$.

Definitions 1.2[8]:

1. The degree of a vertex v in the graph G is the number of edges in G incident with v.
2. A graph G is connected if between every two of its vertices u and v, there is a uv-path.

3. For a connected graph G, the distance d(u, v) between two vertices u and v of G is the minimum of the lengths of the u-v paths of G.
4. The diameter of the connected graph G is the maximum distance between any two distinct vertices in G.
5. The girth of the graph G is the length of the shortest cycle in the graph G that contains a cycle.
6. A complete graph G of order n, denoted by K_n, is graph in which every two of its vertices are adjacent.
7. A bipartite graph is one whose vertex set is partitioned into two disjoint subsets in such a way that the two end vertices for each edge lie in distinct partition. The complete bipartite graph with exactly two partitions of size m and n is denoted by $K_{m, n}$. A graph G is said to be star if G = $K_{1, n}$.
8. A tree is a connected graph which does not contain any cycle.
9. A maximal complete subgraphK_n of a graph G is called a clique, and cl(G) is the clique number of G, which is the greatest integer $r\geq 1$ such that $K_r\subseteq G$.
10. A graph G is called a planar graph if it can be drawn on a plane in such a way that any two of its edges either meet only at their end vertices or do not meet at all.
11. A graph $G_1(V_1, E_1)$ is said to be isomorphic to the graph $G_2(V_2, E_2)$ if there is a one-to-one correspondence between the vertex sets V_1 and V_2 and a one-to-one correspondence between the edge sets E_1 and E_2 in such a way that if e_1 is an edge with end vertices u_1 and v_1 in G_1, then the corresponding edge e_2 in G_2 has its end vertices u_2 and v_2 which corresponds to u_1 and v_1 respectively. Such a pair of correspondence is called a graph isomorphism.
12. A subdivisionof a graph G is a graph obtained from G by removing some edge e=uvand adding new vertex w and edges uw and vw.
13. A graph H is defined to be homeomorphic from G if either H isomorphic to G or H isomorphic to a subdivision of G.

Theorem 1.3[8]:(Kuratowsky Theorem)

A graph G is planar if it does not contain a sub graph homeomorphic to a subdivision of K_5 or $K_{3, 3}$.

Corollary 1.4[1]**:**If n>1 is a composite number, then$\Gamma_n(Z)$ is a finite connected graph with diameter less than or equal to 3, and the girth (if exists) is less than or equal to 4.

2. The Element Ideal Graph $\Gamma_{p^\alpha q^\beta}(Z)$

The main purpose of this section is to explore some properties and characterizations of the element ideal graph $\Gamma_{p^\alpha q^\beta}(Z)$.

Before starting with the results of this section, we give the following example:

Example1:

The graph $\Gamma_{40}(Z)$ is:

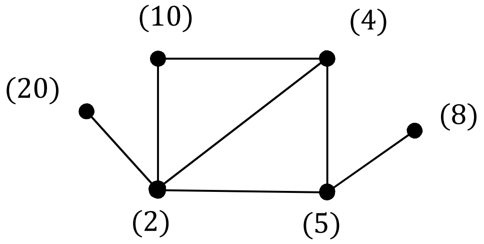

Figure-1: The graph $\Gamma_{40}(Z)$

We start this section with the following Lemma.

Lemma2.1:

The vertex set of$\Gamma_{p^\alpha q^\beta}(Z)$ is $V(\Gamma_{p^\alpha q^\beta}(Z))=\{(x):x|p^\alpha q^\beta$ □ $0<x\neq p^\alpha q^\beta\}$.

Proof:

Let I be an ideal vertex of $\Gamma_{p^\alpha q^\beta}(Z)$. Then there exists $x,y\in Z^+\backslash\{1\}$ with $x\neq y$ such that I=(x) and $p^\alpha q^\beta\in I(y)$. This yields that $p^\alpha q^\beta=rxy$ for some $r\in Z$.Thus$x|p^\alpha q^\beta$.Since p and q are distinct primes then $x=p^i q^j$ where i and j are positive integers with i$\leq \alpha$and $j \leq \beta$.. If we assume that x= $p^\alpha q^\beta$, then $p^\alpha q^\beta$=r$p^\alpha q^\beta$y. This yields that 1=ry∈(y). This contradicts the fact that (y) is a non-trivial ideal of Z. Therefore x≠ $p^\alpha q^\beta$. Hence the vertex set of$\Gamma_{p^\alpha q^\beta}$(Z) is V($\Gamma_{p^\alpha q^\beta}$(Z))={(x):x|$p^\alpha q^\beta$⍰ 0<x≠ $p^\alpha q^\beta$}. □

The graph $\Gamma_{p^\alpha q^\beta}$(Z) contains a complete bipartite subgraph with partite order α and β as the following result shows.

Proposition2.2:

The graph $\Gamma_{p^\alpha q^\beta}$(Z) contains a subgraph isomorphic to $K_{\alpha,\beta}$.

Proof:

Define the Graph G by G={{(p^i), (q^j)}∈E($\Gamma_{p^\alpha q^\beta}$(Z)): i=1, 2,…, α and j=1, 2,…, β}. Since $p^\alpha q^\beta$ ∈(p^i)(q^j) for every i=1, 2,…, α and j=1, 2,…, β, then every element of the set V_1={(p), (p^2), …,(p^α) } is adjacent to every element of the set V_2={(q), (q^2), …,(q^β)}. On the other hand, no elements in V_1 or V_2 are adjacent, since neither $p^\alpha q^\beta$ ∈(p^i)(p^j) nor $p^\alpha q^\beta$ ∈(q^i)(q^j) for every i=1, 2,…, α and j=1, 2,…, β. Thus the graph G is a complete bipartite subgraph of $\Gamma_{p^\alpha q^\beta}$(Z) with partite V_1 and V_2. It is clear from the following figure that G is isomorphic to $K_{\alpha,\beta}$.

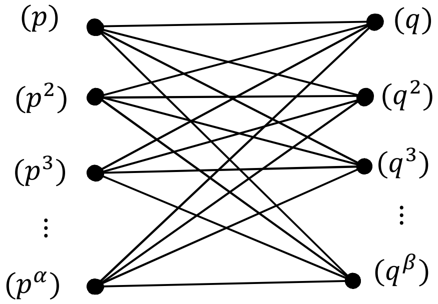

Figure-2: The graph G

□

In the next result, we find the degree of the ideal vertex (p) of $\Gamma_{p^\alpha q^\beta}$(Z).

Proposition2.3:

In the graph $\Gamma_{p^\alpha q^\beta}$(Z), deg((p))= $\alpha+\alpha\beta-2$ and deg((q))= $\beta+\alpha\beta-2$, where $\alpha,\beta>1$.

Proof:

Let $(p^s q^r)$ be an ideal vertex adjacent to (p) in $\Gamma_{p^\alpha q^\beta}$(Z).Then $p^\alpha q^\beta \in (p)(p^s q^r)$. It follows that there exists k∈Z^+ such that $p^\alpha q^\beta = kp^{s+1}q^r$. Thus $p^{s+1}q^r|p^\alpha q^\beta$. This is true when either (s=2, 3 …, $\alpha-1$ and r=0) or (s=0, 1, 2, …, $\alpha-1$ and r=1, 2, …, β) except for possibilities (s, r)=(0, 0) or (s, r)=(α,β). Thus the total number of ideal vertices adjacent to (p) is deg((p))=($\alpha-2$)+$\alpha\beta$= $\alpha+\alpha\beta-2$. Similarly, we can prove that deg((q))= $\beta+\alpha\beta-2$.□

Example2:

Consider the graph $\Gamma_{72}(Z)$. Clearly, 72=$2^3\cdot 3^2$, p=2, q=3, α=3 and β=2.

Thus deg((p))=3+3·2−2=4.

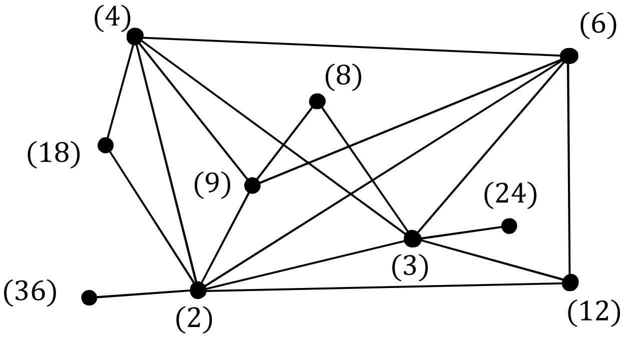

Figure-1: The graph $\Gamma_{72}(Z)$

In the next result, we find the order of the graph $\Gamma_{p^\alpha q^\beta}(Z)$.

Proposition2.4:

Let $\alpha, \beta \in Z^+$The order of the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is equal to $\alpha\beta + \alpha + \beta - 1$.

Proof: We give four cases for α and β:

Case1: If $\alpha=\beta=1$, then the vertex set of $\Gamma_{p^\alpha q^\beta}(Z)$ contains the only ideal vertices (p) and (q). In this case the order of the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is equal to $\alpha\beta + \alpha + \beta - 1=2$.

Case2: Suppose that either $\alpha= 1<\beta$ or $\beta = 1<\alpha$. Let $\alpha= 1 < \beta$. Then by Lemma2.1**,** all ideal vertices of $\Gamma_{pq^\beta}(Z)$ are (p), (pq), (pq^2), …, ($pq^{\beta-1}$) and (q), (q^2), …, (q^β). Thus the order of the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is equal to $2\beta = \alpha\beta + \alpha + \beta - 1$. Similarly, if $\beta = 1<\alpha$, then the order of $\Gamma_{p^\alpha q^\beta}(Z)$ is equal to 2α.

Case3: Let α, $\beta>1$ and let s and rbe two integers such that $0\leq s\leq\alpha$ and $0\leq r\leq \beta$. It is obvious from Lemma3.1 that ($p^s q^r$) is an ideal vertex of $\Gamma_{p^\alpha q^\beta}(Z)$ for every s=0, 1, 2, …, α and r=0, 1, 2, …, β, except for possibility that (s, r)=(0, 0) or (s, r)=(α,β). Thus the order of the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is equal to $(\alpha+1)(\beta+1)-2=\alpha\beta+\alpha+\beta-1$.□

Example3:

Consider the graph $\Gamma_{175}(Z)=\Gamma_{5^2.7}(Z)$. We have α=2 and β=1.

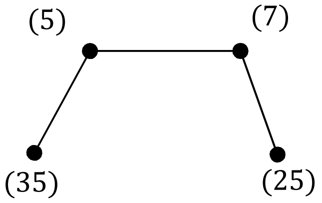

Figure-4: The graph $\Gamma_{175}(Z)$

Clearly, the order of $\Gamma_{175}(Z)$ is equal to $\alpha\beta + \alpha + \beta - 1 = 4$.

The next result demonstrates the incompleteness of the graph $\Gamma_{p^\alpha q^\beta}(Z)$ under a certain condition.

Theorem2.5:

If G is a complete graph of order greater than 2, then G cannot be realized as an element ideal graph of the form $\Gamma_{p^\alpha q^\beta}(Z)$.

Proof:

Suppose that G is a complete graph of order greater than 2 and G=$\Gamma_{p^\alpha q^\beta}(Z)$ for somedistinct prime numbers p and q, and $\alpha,\beta\in Z^+$. Clearly, at least one of α and β is greater than 1.Suppose that$\alpha>1$. Since $p^\alpha q^\beta \in (p)(q) \cap (p^\alpha)(q)$, then (p) and (p^α) are ideal vertices in $\Gamma_{p^\alpha q^\beta}(Z)$. But $p^\alpha q^\beta \notin (p)(p^\alpha)$, so (p) and (p^α) are not adjacent in $\Gamma_{p^\alpha q^\beta}(Z)$. This contradicts the fact that G is a complete graph. Therefore G can not be realized as an element ideal graph of the form $\Gamma_{p^\alpha q^\beta}(Z)$.□

In [1], the diameter of the element ideal graph has been proved that less than or equal to 3. The following resultshows that the diameter of$\Gamma_{p^\alpha q^\beta}(Z)$ is equal to 3.

Theorem2.6:

If $\alpha,\beta\in Z^+$ such that the order of the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is greater than 2, then the diameter of$\Gamma_{p^\alpha q^\beta}(Z)$ is equal to 3.

Proof: Since the order of the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is greater than 2, then at least one of α and β is greater than one. Suppose that $\beta>1$. If α=1, then P_4:($pq^{\beta-1}$), (q), (p), (q^β) is the shortest path between ($pq^{\beta-1}$) and (q^β) in $\Gamma_{p^\alpha q^\beta}(Z)$ of length 3. Thus diam($\Gamma_{p^\alpha q^\beta}(Z)$)≥3. It follows from Corollary1.4 that diam($\Gamma_{p^\alpha q^\beta}(Z)$)=3.

$\alpha>1$, then P_4:($p^{\alpha-1}q^{\beta}$), (p), (q), ($p^{\alpha}q^{\beta-1}$) is the shortest path between ($p^{\alpha-1}q^{\beta}$) and ($p^{\alpha}q^{\beta-1}$) in $\Gamma_{p^{\alpha}q^{\beta}}(Z)$ of length 3. It follows from Corollary1.4 that diam($\Gamma_{p^{\alpha}q^{\beta}}(Z)$)=3. By the same way we can prove that diam($\Gamma_{p^{\alpha}q^{\beta}}(Z)$)=3, if $\alpha>1$ and $\beta=1$.□

Example4:

Consider the graph $\Gamma_{2^3.3}(Z)$.

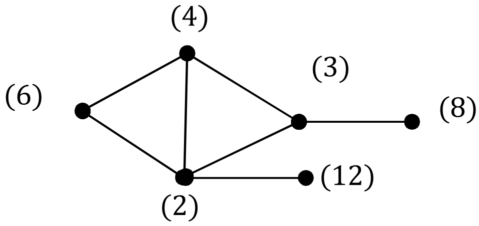

Figure-5: The graph $\Gamma_{2^3.3}(Z)$

Obviously, the diameter of $\Gamma_{2^3.3}(Z)$ is equal to 3.

Remark2.7:

If $\alpha=\beta=1$, then the graph $\Gamma_{p^{\alpha}q^{\beta}}(Z)$ consists of the only edge {(p), (q)}, so $\Gamma_{p^{\alpha}q^{\beta}}(Z)$ is a complete graph and its diameter is equal to 1.

The next result determines the planarity of the graph $\Gamma_{p^{\alpha}q^{\beta}}(Z)$under a condition of α and β.

Theorem2.8:

Let $\alpha, \beta \in Z^+$ with $\alpha \leq \beta$. Then the graph $\Gamma_{p^{\alpha}q^{\beta}}(Z)$ is planar if $\alpha+\beta<6$, otherwise$\Gamma_{p^{\alpha}q^{\beta}}(Z)$ is non-planar.

Proof:

Let $\alpha+\beta<6$. If $\alpha=1$, then $\beta\leq4$. The graph $\Gamma_{p^1q^4}(Z)$ is:

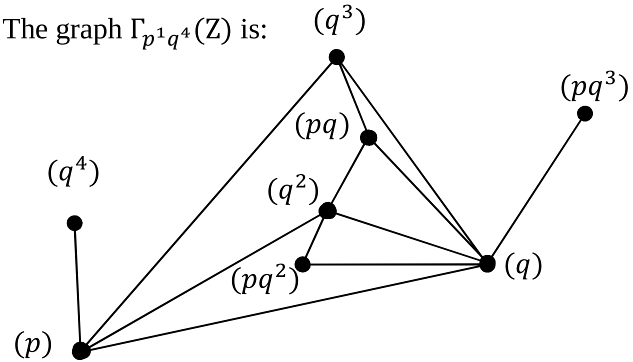

Figure-6: The graph $\Gamma_{p^1q^4}(Z)$

It is clear from Figure-6 that $\Gamma_{p^1q^4}(Z)$ is planar. Since the graphs $\Gamma_{p^1q^1}(Z)$, $\Gamma_{p^1q^2}(Z)$ and $\Gamma_{p^1q^3}(Z)$ are subgraphs of $\Gamma_{p^1q^4}(Z)$, then the graphs $\Gamma_{p^1q^{\beta}}(Z)$is aplanar graph for every $\beta\leq4$.

If $\alpha=2$, then $\in\{2, 3\}$. The graph $\Gamma_{p^2q^3}(Z)$ is:

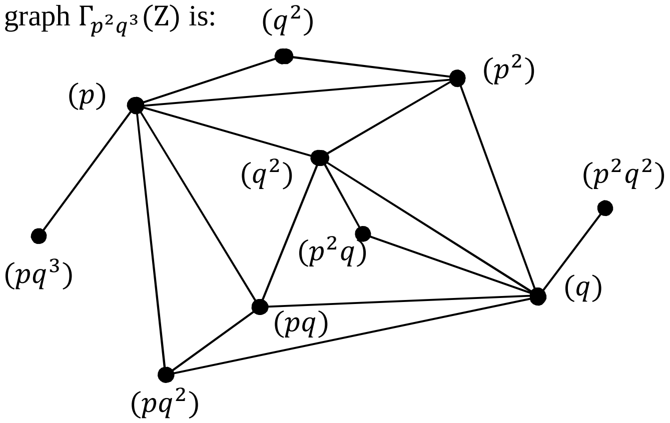

Figure-7: The graph$\Gamma_{p^2q^3}(Z)$

It is obvious from Figure-7 that$\Gamma_{p^2q^3}(Z)$ is planar graph. Since the graph $\Gamma_{p^2q^2}(Z)$is a subgraphof $\Gamma_{p^2q^3}(Z)$, then $\Gamma_{p^2q^2}(Z)$ is also planar.

Suppose that $\alpha+\beta\geq 6$. We have three cases for α:

Case1: If α=1, then β>4. Thus the graph $\Gamma_{p^1q^\beta}(Z)$ contains complete bipartite graph $K_{3,3}$ of partite sets {(p), (pq), (pq^2)} and {(q), (q^2), (q^3)}.

Case2: If α=2, then β>3. Thus the graph $\Gamma_{p^2q^\beta}(Z)$ contains complete bipartite graph $K_{3,3}$ of partite sets {(p), (p^2), (pq)} and {(q), (q^2), (q^3)}.

Case3: If $\alpha\geq 3$, then β>3. Thus the graph $\Gamma_{p^\alpha q^\beta}(Z)$ contains complete bipartite graph $K_{3,3}$ of partite sets {(p), (p^2), (p^3)} and {(q), (q^2), (q^3)}.

The above three cases and the Koratowsky Theorem gives that the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is non-planar.□

3. The Clique number of $\Gamma_{p^\alpha q^\beta}(Z)$

This section is devoted to find the clique number and the girth of the graph $\Gamma_{p^\alpha q^\beta}(Z)$.

Theorem3.1:

The clique number of the graph $\Gamma_{pq^\beta}(Z)$ is cl($\Gamma_{pq^\beta}(Z)$)=$\begin{cases} \frac{\beta}{2}+1, & if\ \beta\ is\ even \\ \frac{\beta+1}{2}+1 & if\ \beta\ is\ odd \end{cases}$.

Proof:

Suppose that β is an even number. Then any two of ideal vertices (p), (q), (q^2), …,($q^{\frac{\beta}{2}}$) are adjacent in $\Gamma_{pq^\beta}(Z)$ whose number is $\frac{\beta}{2}+1$. Thus$\Gamma_{pq^\beta}(Z)$ contains a complete subgraph G of vertices(p), (q), (q^2), …,($q^{\frac{\beta}{2}}$). To show that G is a maximal complete subgraph of $\Gamma_{pq^\beta}(Z)$, let I be an ideal vertex adjacent to all vertices (p), (q), (q^2), …,($q^{\frac{\beta}{2}}$). Since I is adjacent to (p), then p is not a coefficient of the generator of I. Thus I=(q^s) for some s∈Z$^+$. Since I=(q^s) is adjacent to ($q^{\frac{\beta}{2}}$), then $pq^\beta\in(q^s)((q^{\frac{\beta}{2}}))$.This implies that s≤ $\frac{\beta}{2}$. On other side I=(q^s) is different from every ideal vertices (q), (q^2), …,($q^{\frac{\beta}{2}}$). Thus s>$\frac{\beta}{2}$.Which is impossible. Therefore G is a maximal complete subgraph of $\Gamma_{pq^\beta}(Z)$ of order $\frac{\beta}{2}+1$. Hence cl($\Gamma_{pq^\beta}(Z)$)=$\frac{\beta}{2}+1$. By the same way we can prove that cl($\Gamma_{pq^\beta}(Z)$)=$\frac{\beta+1}{2}+1$, if β is an odd number. □

Theorem3.2:

Let $\alpha, \beta\in Z^+$. Then the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is a tree if and only if$\alpha+\beta<4$.

Proof:

Suppose that $\alpha+\beta<4$. Then wehave three cases for α and β:

Case1: If $\alpha=\beta=1$, then the graph $\Gamma_{p^\alpha q^\beta}(Z)$ consists of the only edge {(p),(q)}. In this case $\Gamma_{p^\alpha q^\beta}(Z)$ is a tree.

Case2: If α=1 and β=2, then the graph $\Gamma_{p^\alpha q^\beta}(Z)$ consists of a path P_4:(q^2), (p), (q), (pq). In this case the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is a tree.

Case3: If α=2 and β=1, then by the same concept, $\Gamma_{p^\alpha q^\beta}(Z)$ will be a tree.

Conversely, ifwe assume that$\alpha+\beta\geq 4$, then either α>1 or β>1. If α>1, then C_3:(p), (pq), (q), (p) is a cycle in $\Gamma_{p^\alpha q^\beta}(Z)$. If β>1, then C_3:(q), (pq), (p), (q) is a cycle in $\Gamma_{p^\alpha q^\beta}(Z)$. In both subcases the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is not tree.□

Corollary3.3:

The girth of the graph $\Gamma_{p^\alpha q^\beta}(Z)$ is g($\Gamma_{p^\alpha q^\beta}(Z)$)=$\begin{cases} \infty, & if \alpha+\beta<4, \\ 3, & if\ \alpha+\beta\geq 4 \end{cases}$.

Proof:

If $\alpha+\beta<4$, then by Theorem3.2 the graph $\Gamma_{p^\alpha q^\beta}(Z)$ does not contain any cycle. Thus the girth $\Gamma_{p^\alpha q^\beta}(Z)$of is equal to ∞. If $\alpha+\beta\geq 4$, then by Theorem3.2 the graph $\Gamma_{p^\alpha q^\beta}(Z)$ contains a cycle of length three.

Thus the girth of $\Gamma_{p^\alpha q^\beta}(\mathrm{Z})$ is equal to 3.□

Example5:

In the graph $\Gamma_{6875}(\mathrm{Z})$=$\Gamma_{11\cdot 5^4}(\mathrm{Z})$, we have α=1, β=2n=4. Thus n=2.

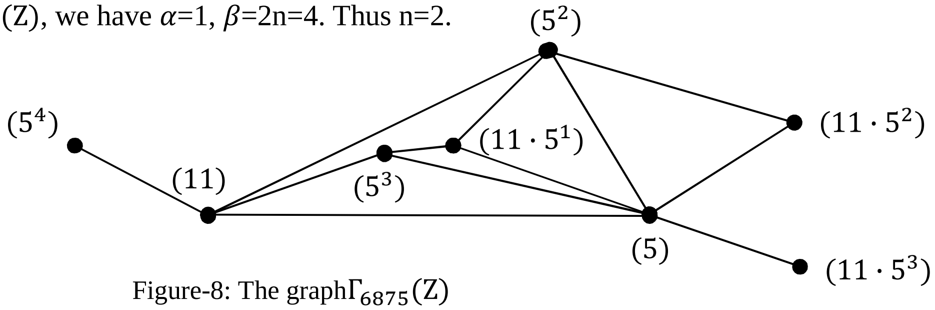

Figure-8: The graph$\Gamma_{6875}(\mathrm{Z})$

Clearly, cl($\Gamma_{6875}(\mathrm{Z})$)=2+1=3.
Since $\alpha + \beta = 5 \geq 4$, then g($\Gamma_{6875}(\mathrm{Z})$)=3.

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